ISRAEL JOURNAL OF MATHEMATICS 93 (1996), 1–8

THE ESSENTIALLY FREE SPECTRUM OF A VARIETY

BY

ALAN H. MEKLER*

Department of Mathematics and Statistics
Simon Fraser University, Burnaby, B.C. V5A 1S6 Canada

AND

SAHARON SHELAH**

Institute of Mathematics, The Hebrew University of Jerusalrm
Givat Ram, 91904 Jerusalem, Israel
e-mail: shelah@math.huji.ac.il

AND

OTMAR SPINAS***

Department of Mathematics
University of California, Irvine, CA 92717, USA
e-mail: ospinas@math.uci.edu

ABSTRACT

We partially prove a conjecture from [MeSh] which says that the spectrum of almost free, essentially free, non-free algebras in a variety is either empty or consists of the class of all successor cardinals.

* The author was supported by the NSERC. Prof. Mekler died on June 10, 1992.

** The author is supported by the United States–Israel Binational Science Foundation; publication 417.

*** The author is supported by the Schweizer Nationalfonds.

Received June 29, 1994

Introduction and notation

Suppose that T is a **variety** in a countable vocabulary τ. This means that τ is a countable set of function symbols and T is a set of equations, i.e. sentences of the form $\forall x_1, \ldots, x_n\ (\sigma_1(x_1, \ldots, x_n) = \sigma_2(x_1, \ldots, x_n))$ where σ_i are τ-terms. The class of all models of T will be denoted by $\mathrm{Mod}(T)$, and a member of $\mathrm{Mod}(T)$ is called an **algebra** in the variety T. Let $M \in \mathrm{Mod}(T)$. For $A \subseteq M$, $\langle A \rangle$ denotes the submodel of M generated by A. Such A is called a **free basis** (of $\langle A \rangle$) if no distinct $a_1, \ldots, a_n \in A$ satisfy an equation which is not provable from T. Moreover, M is called **free** if there exists a free basis of M, i.e. one which generates M. By F_λ we denote the free algebra with free basis of size λ, where λ is a cardinal. For $M_1, M_2 \in \mathrm{Mod}(T)$, the **free product** of M_1 and M_2 is denoted by $M_1 * M_2$. Formally it is obtained by building all formal terms in the language τ with constants belonging to the disjoint union of M_1 and M_2, and then identifying them according to the laws in T. For M, $\langle M_\nu \colon \nu < \alpha \rangle$ such that $M, M_\nu \in \mathrm{Mod}(T)$ and M is a submodel of M_ν for all $\nu < \alpha$, the **free product of the M_ν's over M** is defined similarly, and it is denoted by $*_M\{M_\nu \colon \nu < \alpha\}$; the intention being that distinct M_ν, $M_{\nu'}$ are disjoint outside M except for those equalities which follow from the laws in T and the equations in $\mathrm{Diag}(M_\nu) \cup \mathrm{Diag}(M_{\nu'})$. Here Diag denotes the **diagram** of a model. For $M, N \in \mathrm{Mod}(T)$ we say "N/M is **free**" if M is a submodel of N and there exists a free basis A **of N over M**, i.e. A is a free basis, $N = \langle M \cup A \rangle$ and between members of $\langle A \rangle$ and M only those equations hold which follow from T and $\mathrm{Diag}(M)$.

Suppose $|M| = \lambda$. Then M is called **almost free** if there exists an increasing continuous family $\langle M_\nu \colon \nu < \mathrm{cf}(\lambda) \rangle$ of free submodels of size $< \lambda$ with union M. Moreover, M is called **essentially free** if there exists a free $M' \in \mathrm{Mod}(T)$ such that $M * M'$ is free, **essentially non-free** otherwise. The essentially free spectrum of the variety T which is denoted by $\mathrm{EINC}(T)$, is the class of cardinals λ such that there exists $M \in \mathrm{Mod}(T)$ of size λ which is almost free and essentially free, but not free.

In [MeSh] the **essentially non-free spectrum**, i.e. the spectrum of cardinalities of almost free and essentially non-free algebras in a variety T, has been investigated, and it is shown that this spectrum has no simple description in ZFC, in general. Here we will show that the situation is different for $\mathrm{EINC}(T)$. Firstly, by a general compactness theorem due to the second author (see [Sh]), $\mathrm{EINC}(T)$ contains only regular cardinals. Secondly, we will show that $\mathrm{EINC}(T)$

is contained in the class of successor cardinals. Our conjecture is that EINC(T) is either empty or equals the class of all successor cardinals (depending on T). Motivating examples for this conjecture are among others $\mathbb{Z}/4\mathbb{Z}$-modules (where EINC is empty) and $\mathbb{Z}/6\mathbb{Z}$-modules (where EINC consists of all successor cardinals) [EkMe, p.90]. We succeed to prove the conjecture to a certain extent. Namely, we prove the following theorem.

THEOREM: *If for some cardinal* μ, $(\mu^{\aleph_0})^+ \in \mathrm{EINC}(T)$, *then every successor cardinal belongs to* EINC(T).

For the proof we will isolate a property of T, denoted $\mathrm{Pr}_1(T)$, which says that a countable model of T with certain properties exists, and then show that, on the one hand, the existence of $M \in \mathrm{Mod}(T)$ in any cardinality of the form $(\mu^{\aleph_0})^+$ implies that $\mathrm{Pr}_1(T)$ holds, and on the other hand, from $\mathrm{Pr}_1(T)$ an algebra $M \in \mathrm{Mod}(T)$ can be constructed in every successor cardinality.

1. EINC(T) is contained in the class of successor cardinals

THEOREM: *For every variety* T, EINC(T) *is contained in the class of successor cardinals.*

Proof: Suppose $\lambda \in \mathrm{EINC}(T)$. By the main result of [Sh], λ must be regular. So suppose λ is a regular limit cardinal. Let $\mathfrak{M} \in \mathrm{Mod}(T)$ be generated by $\{a_\alpha\colon \alpha < \lambda\}$ and suppose that $\mathfrak{M}$ is almost free and essentially free. We will show that then $\mathfrak{M}$ must be free, and hence does not exemplify $\lambda \in \mathrm{EINC}(T)$.

By assumption and a Löwenheim–Skolem argument, $\mathfrak{M} * F_\lambda$ is free. Let $\{c_\nu\colon \nu < \lambda\}$, $\{b_\nu\colon \nu < \lambda\}$ be a free basis of $\mathfrak{M} * F_\lambda$, F_λ, respectively.

Let χ be a large enough regular cardinal, and let $C \subseteq \lambda$ be the club consisting of all α such that for some substructure $\mathcal{A} \prec \langle H(\chi), \in, \prec_\chi \rangle$ of size $< \lambda$ which contains $\mathfrak{M}$, F_λ, $\{a_\nu\colon \nu < \lambda\}$, $\{b_\nu\colon \nu < \lambda\}$ and $\{c_\nu\colon \nu < \lambda\}$, we have $\mathcal{A} \cap \lambda = \alpha$. Here $H(\chi)$ is the set of all sets which are hereditarily of cardinality $< \chi$, and $\prec_\chi$ is a fixed well-ordering of $H(\chi)$. Note that the information about $\mathfrak{M}$ reflects to each $\alpha \in C$, especially $\langle\{c_\nu\colon \nu < \alpha\}\rangle = \langle\{a_\nu\colon \nu < \alpha\}\rangle * \langle\{b_\nu\colon \nu < \alpha\}\rangle$.

Since $\mathfrak{M}$ is supposed to be almost free, the set

$$C_0 = \{\alpha \in C\colon \langle\{a_\nu\colon \nu < \alpha\}\rangle \text{ is free}\}$$

is still a club. Let $\alpha, \beta \in C_0$ be cardinals with $\alpha < \beta$. We will show that $\langle\{a_\nu\colon \nu < \beta\}\rangle / \langle\{a_\nu\colon \nu < \alpha\}\rangle$ is free. This will suffice to conclude that $\mathfrak{M}$ is free since the cardinals below λ are a club and hence $C_1 = \{\alpha \in C_0\colon \alpha \text{ is a cardinal}\}$ is a club such that for every $\alpha, \beta \in C_1$ with $\alpha < \beta$, $\langle\{a_\nu\colon \nu < \beta\}\rangle / \langle\{a_\nu\colon \nu < \alpha\}\rangle$ is free.

For the proof, let $\{d_\nu\colon \nu < \beta\}$ be a free basis of $\langle\{a_\nu\colon \nu < \beta\}\rangle$. As $\alpha = |\alpha| < |\beta| = \beta$ we may assume $\langle\{a_\nu\colon \nu < \alpha\}\rangle \subseteq \langle\{d_\nu\colon \nu < \alpha\}\rangle$. Hence easily

$$\langle\{a_\nu\colon \nu < \beta\}\rangle \cong_{\langle\{a_\nu\colon \nu<\alpha\}\rangle} \langle\{a_\nu\colon \nu < \beta\}\rangle * F_\beta,$$

i.e. there exists an isomorphism which leaves $\langle\{a_\nu\colon \nu < \alpha\}\rangle$ fixed. But $\langle\{a_\nu\colon \nu < \beta\}\rangle * F_\beta \cong \langle\{c_\nu\colon \nu < \beta\}\rangle$ and $\langle\{c_\nu\colon \nu < \beta\}\rangle / \langle\{c_\nu\colon \nu < \alpha\}\rangle$ is free. Moreover $\langle\{c_\nu\colon \nu < \alpha\}\rangle = \langle\{a_\nu\colon \nu < \alpha\}\rangle * \langle\{b_\nu\colon \nu < \alpha\}\rangle$ and hence $\langle\{c_\nu\colon \nu < \alpha\}\rangle / \langle\{a_\nu\colon \nu < \alpha\}\rangle$ is free. Consequently $\langle\{a_\nu\colon \nu < \beta\}\rangle / \langle\{a_\nu\colon \nu < \alpha\}\rangle$ is free. ∎

2. EINC(T) is either empty or contains almost all successor cardinals

Definition 2.1: The property $\mathrm{Pr}_1(T)$ says: There exist $N, M \in \mathrm{Mod}(T)$ such that N is countably generated, M is a subalgebra of N and the following clauses hold:

(i) M has a free basis;

(ii) $N * F_{\aleph_0}/M$ is free;

(iii) $*_M\{N\colon n \in \omega\} * F_{\aleph_0}/M * F_{\aleph_0}$ is not free.

THEOREM 2.2: *Suppose that $Pr_1(T)$ holds and λ is a successor cardinal. Then $\lambda \in \mathrm{EINC}(T)$.*

Proof: Let $\lambda = \mu^+$. Let M, N witness $\mathrm{Pr}_1(T)$. Let $\mathfrak{N} = *_M\{N\colon \alpha < \lambda\}$. We claim that $\mathfrak{M} = \mathfrak{N} * F_\mu$ exemplifies that $\lambda \in \mathrm{EINC}(T)$. Let $\{c_\alpha\colon \alpha < \mu\}$ be a free basis of F_μ.

Firstly, $\mathfrak{M}$ is almost free: For $\alpha < \lambda$ let $\mathfrak{N}_\alpha = *_M\{N\colon \nu < \alpha\}$. Then clearly $\langle \mathfrak{N}_\alpha * F_\mu\colon \alpha < \lambda\rangle$ is a λ-filtration of $\mathfrak{M}$. Moreover $\mathfrak{N}_\alpha * F_\mu$ is free for every $\alpha < \lambda$, since easily $\mathfrak{N}_\alpha * F_\mu \cong *_M\{N * F_{\aleph_0}\colon \nu < \alpha\}$ and by $\mathrm{Pr}_1(T)$, M is free and $N * F_{\aleph 0}/M$ is free.

Secondly, $\mathfrak{M} * F_\lambda \cong \mathfrak{N} * F_\lambda$ is free, since $\mathfrak{N} * F_\lambda \cong *_M\{N * F_{\aleph_0}\colon \alpha < \lambda\}$ is free as in the proof of almost freeness.

Thirdly, $\mathfrak{M}$ is not free. By contradiction, suppose that $I = \{d_\nu\colon \nu < \lambda\}$ were a free basis of $\mathfrak{M}$.

Let χ be a large enough regular cardinal, and let $\mathcal{A} \prec \langle H(\chi), \in, \prec_\chi \rangle$ such that $|\mathcal{A}| = \mu$, $\mu + 1 \subseteq \mathcal{A}$, and $\lambda^+, N, M, \mathfrak{N}, \mathfrak{M}, F_\mu, I \in \mathcal{A}$. Next choose $\mathcal{B} \prec \langle H(\chi), \in, \prec_\chi \rangle$ such that $|\mathcal{B}| = \aleph_0$, and $\mathcal{A}, \lambda^+, N, M, \mathfrak{N}, \mathfrak{M}, F_\mu, I \in \mathcal{B}$.

Let $u = \mathcal{B} \cap \lambda \setminus (\mathcal{A} \cap \lambda)$, $v = \mathcal{A} \cap \mathcal{B} \cap \lambda$, $w = \mathcal{A} \cap \mathcal{B} \cap \mu$. Notice that $w = \mathcal{B} \cap \mu$. Define $M_1 = \mathcal{A} \cap \mathcal{B} \cap \mathfrak{M}$. Now easily M_1 is countably generated and it has the form

$$M_1 = *_M\{N: \alpha \in v\} * \langle \{c_\alpha: \alpha \in w\} \rangle.$$

Hence $M_1 \cong_M *_M\{N: n \in \omega\} * F_{\aleph_0} \cong_M *_M\{N * F_{\aleph_0}: n \in \omega\} \cong_M M * F_{\aleph_0}$, where for the last isomorphy we applied (ii) from $\mathrm{Pr}_1(T)$. Next define $M_2 = \mathcal{B} \cap \mathfrak{M}$. Then easily

$$M_2 = *_M\{N: \alpha \in u\} *_M M_1.$$

Hence by the isomorphy above we have

$$M_2 \cong *_M\{N: n \in \omega\} * F_{\aleph_0}.$$

By (iii) from $\mathrm{Pr}_1(T)$ we conclude that M_2/M_1 is not free. On the other hand, $\{d_\nu: \nu \in u\}$ witnesses that M_2/M_1 is free, a contradiction. ∎

THEOREM 2.3: *Suppose λ, μ are cardinals such that $\lambda = \mu^+$, $\mu^{\aleph_0} = \mu$ and $\lambda \in \mathrm{EINC}(T)$. Then $Pr_1(T)$ holds.*

Proof: Let $\mathfrak{M}$ exemplify $\lambda \in \mathrm{EINC}(T)$. Let $\{a_\nu: \nu < \lambda\}$ generate $\mathfrak{M}$. Let F be free such that $\mathfrak{M} * F$ is free. Without loss of generality we may assume that $F = F_\lambda$; in fact, if $|F| < \lambda$ then we may replace F by $F * F_\lambda$ which is isomorphic to F_λ, and if $|F| > \lambda$ use a Löwenheim–Skolem argument. So let $\{b_\nu: \nu < \lambda\}$ be a free basis of F, and let $\{c_\nu: \nu < \lambda\}$ be a free basis of $\mathfrak{N} = \mathfrak{M} * F$.

Let χ be a large enough regular cardinal, and let N_α, for every $\alpha < \lambda$, be a countable elementary substructure of $\langle H(\chi), \in, \prec_\chi \rangle$ such that $\alpha, \mathfrak{M}, F, \mathfrak{N}$, $\{a_\nu: \nu < \lambda\}, \{b_\nu: \nu < \lambda\}, \{c_\nu: \nu < \lambda\}$ belong to N_α. Let $u_\alpha = N_\alpha \cap \lambda$.

By assumption on $\mathfrak{M}$ ($\mathfrak{M}$ is almost free), the set

$$\{\alpha < \lambda: \langle \{a_\nu: \nu < \alpha\} \rangle \text{ is free } \wedge \langle \{c_\nu: \nu < \alpha\} \rangle = \langle \{a_\nu: \nu < \alpha\} \rangle * \langle \{b_\nu: \nu < \alpha\} \rangle\}$$

contains a club; let C be the $\prec_\chi$ –least one. Hence $C \in N_\alpha$ for every $\alpha < \lambda$.

Using elementarity, it is easy to see that for every $\alpha \in C$ the following three clauses hold ((1) holds for every $\alpha < \lambda$):

(1) $\langle \{c_\nu: \nu \in u_\alpha\} \rangle = \langle \{a_\nu: \nu \in u_\alpha\} \rangle * \langle \{b_\nu: \nu \in u_\alpha\} \rangle$;

(2) $\langle\{c_\nu : \nu \in u_\alpha \cap \alpha\}\rangle = \langle\{a_\nu : \nu \in u_\alpha \cap \alpha\}\rangle * \langle\{b_\nu : \nu \in u_\alpha \cap \alpha\}\rangle$;

(3) $\{\langle a_\nu : \nu \in u_\alpha \cap \alpha\}\rangle$ is free, and $\langle\{a_\nu : \nu \in \alpha\}\rangle / \{\langle a_\nu : \nu \in u_\alpha \cap \alpha\}\rangle$ is free.

To prove (3), let $\langle d_\nu : \nu \in I\rangle$ be the $\prec_\chi$ −least free basis of $\langle\{a_\nu : \nu \in \alpha\}\rangle$. So by elementarity $\langle d_\nu : \nu \in I\rangle \in N_\alpha$ and $\langle\{d_\nu : \nu \in I \cap N_\alpha\}\rangle = \langle\{a_\nu : \nu \in u_\alpha \cap \alpha\}\rangle$. Hence $\{d_\nu : \nu \in I \cap N_\alpha\}$ and $\{d_\nu : \nu \in I \setminus N_\alpha\}$ witness that (3) holds.

Moreover it is not difficult to see that $C_0 = \{\alpha \in C : \alpha = \bigcup\{u_\nu : \nu < \alpha\}\}$ is still a club. Hence $S_0 = \{\alpha \in C_0 : (\alpha) > \omega\}$ is stationary. By Fodor's Lemma, for some $\alpha^* < \lambda$, $S_1 = \{\alpha \in S_0 : u_\alpha \cap \alpha \subseteq \alpha^*\}$ is stationary. By assumption, $|\alpha^*|^{\aleph_0} \leq \mu^{\aleph_0} < \lambda$. So by thinning out S_1 further (using this assumption and the λ−completeness of the nonstationary ideal on λ), we may find a stationary $S_2 \subseteq S_1$ and $u^* \subseteq \alpha^*$ such that for every $\delta_1, \delta_2 \in S_2$ the following hold:

(4) $u_{\delta_1} \cap \delta_1 = u^*$;

(5) $\mathrm{o.t.}(u_{\delta_1}) = \mathrm{o.t.}(u_{\delta_2})$, and the unique order-preserving map $h = h_{\delta_1\delta_2} : u_{\delta_1} \to u_{\delta_2}$ induces (by $c_\nu \to c_{h(\nu)}$) an isomorphism from $\langle\{c_\nu : \nu \in u_{\delta_1}\}\rangle$ onto $\langle\{c_\nu : \nu \in u_{\delta_2}\}\rangle$ which maps a_ν to $a_{h(\nu)}$ and b_ν to $b_{h(\nu)}$.

Let $\delta^* = \min(S_2 \setminus \mu)$, $M = \langle\{a_\nu : \nu \in u^*\}\rangle$ and $N = \langle\{a_\nu : \nu \in u_{\delta^*}\}\rangle$.

As $\delta^* \in C$, by elementarity we know that M is free.

As $\{c_\nu : \nu \in \lambda\}$ is a free basis, clearly $\langle\{c_\nu : \nu \in u_{\delta^*}\}\rangle / \langle\{c_\nu : \nu \in u^*\}\rangle$ is free, and by (2) and as $\delta^* \in S_2 \subseteq C$, also $\langle\{c_\nu : \nu \in u^*\}\rangle / M$ is free. Finally, $\langle\{c_\nu : \nu \in u_{\delta^*}\}\rangle \cong N * F_{\aleph_0}$ by (1). Hence we conclude that $N * F_{\aleph_0} / M$ is free.

Hence, if the pair M, N does not exemplify $\mathrm{Pr}_1(T)$, then (iii) in its definition fails. We will use this to show that then $\mathfrak{M}$ is free, which contradicts our assumption. Then we conclude that $\mathrm{Pr}_1(T)$ holds.

By induction on $\zeta < \lambda$ we choose $w_\zeta \subseteq \lambda$ such that the following requirements are satisfied:

(6) $w_0 = \delta^*$;

(7) $|w_\zeta| < \lambda$;

(8) for ζ limit, $w_\zeta = \bigcup\{w_\nu : \nu < \zeta\}$;

(9) if $\gamma(\zeta) = \min(\lambda \setminus w_\zeta)$, then $w_{\zeta+1} = w_\zeta \cup \{\gamma(\zeta)\} \cup \{\beta(\zeta, n) : n \in \omega\}$, where the $\beta(\zeta, n)$ belong to S_2, and for any $m, n \in \omega$ with $m < n$, $\bigcup\{u_{\gamma(\zeta)}, u_\nu : \nu \in w_\zeta\} < \min(u_{\beta(\zeta,n)} \setminus u^*)$ and $\sup(u_{\beta(\zeta,m)}) < \min(u_{\beta(\zeta,n)} \setminus u^*)$ hold.

By (6) and $\delta^* \in C_0 \subseteq C$ we conclude that $\bigcup\{u_\nu : \nu \in w_0\} = \delta^*$ and $\langle\{a_\alpha : \alpha \in \delta^*\}\rangle$ is free. By (8) and (9) it is clear that the sequence

$$\langle\langle\{a_\alpha : \alpha \in \bigcup\{u_\nu : \nu \in w_\zeta\}\}\rangle : \zeta < \lambda\rangle$$

is increasing and continuous with limit $\mathfrak{M}$. Hence the following claim gives the desired contradiction:

CLAIM: *For every $\zeta < \lambda$,*

$$\langle\{a_\alpha: \alpha \in \bigcup\{u_\nu: \nu \in w_{\zeta+1}\}\}\rangle / \langle\{a_\alpha: \alpha \in \bigcup\{u_\nu: \nu \in w_\zeta\}\}\rangle$$

is free.

Proof: Let us introduce the following notation. For $x \in \{a, b, c\}$ and $I \subseteq \lambda$ set:

$$\begin{aligned} Z_I^x &= \langle\{x_\alpha: \alpha \in \bigcup\{u_\nu: \nu \in I\}\}\rangle, \\ W_\zeta^x &= Z_{w_\zeta}^x, \\ K^x &= \langle\{x_\alpha: \alpha \in u^*\}\rangle, \text{ so } K^a = M. \end{aligned}$$

The Claim will follow from the following three facts:

(10) $Z_I^c = Z_I^a * Z_I^b$;

(11) $\langle W_\zeta^a \cup Z_{\{\gamma(\zeta)\}}^a\rangle * F_{\aleph_0}/W_\zeta^a$ is free;

(12) $W_{\zeta+1}^a = *_{K^a}\{Z_{w_\zeta\cup\{\gamma(\zeta)\}}^a, Z_{\{\beta(\zeta,n)\}}^a: n \in \omega\}$.

For (10), to prove $Z_I^c = \langle Z_I^a \cup Z_I^b\rangle$ is rather straightforward by using (1). Moreover, there exists a homomorphism $h: \mathfrak{M} * F \to Z_I^c$ which is the identity on Z_I^c and maps $\mathfrak{M}$ onto Z_I^a; h can be defined by letting

$$h(c_\alpha) = \begin{cases} c_\alpha & \text{if } \alpha \in \bigcup\{u_\nu: \nu \in I\}, \\ a_0 & \text{otherwise.} \end{cases}$$

Now suppose that $\langle Z_I^a \cup Z_I^b\rangle \models \phi(\bar{a}, \bar{b})$, where ϕ is an equation and $\bar{a} \subseteq \{a_\alpha: \alpha \in \bigcup\{u_\nu: \nu \in I\}\}$, $\bar{b} \subseteq \{b_\alpha: \alpha \in \bigcup\{u_\nu: \nu \in I\}\}$ are finite. Then this equation holds in $\mathfrak{M} * F$, of course. As $\{b_\nu: \nu < \lambda\}$ is a free basis of F, we conclude that $\phi(\bar{a}, \bar{b})$ is provable from finitely many equations in Diag($\mathfrak{M}$) and the laws of the variety. But h maps this proof to a proof from Diag(Z_I^a) and leaves $\bar{a}, \bar{b}$ fixed. Consequently $Z_I^c = Z_I^a * Z_I^b$ holds.

To prove (11), first clearly $Z_{w_\zeta\cup\{\gamma(\zeta)\}}^c / W_\zeta^c$ is free. Hence by (10), $Z_{w_\zeta\cup\{\gamma(\zeta)\}}^c / W_\zeta^a$ is free. We may assume that $|w_\zeta| = \mu$; hence $Z_{w_\zeta\cup\{\gamma(\zeta)\}}^a * F_\mu / W_\zeta^a$ is free. How to get μ down to $\aleph_0$? Choose $\bar{N} \prec (H(\chi), \in, \prec_\chi)$ such that $\bar{N}$ is countable and contains everything relevant, especially W_ζ^a, $Z_{\gamma(\zeta)}^a$, F_μ. Let $X \in \bar{N}$ be a free basis of $Z_{w_\zeta\cup\{\gamma(\zeta)\}}^a * F_\mu$ over W_ζ^a. Then $X \cap \bar{N}$ is a free basis of $\bar{N} \cap (Z_{w_\zeta\cup\{\gamma(\zeta)\}}^a * F_\mu)$ over $\bar{N} \cap W_\zeta^a$. Moreover

$$\begin{aligned} \bar{N} \cap (Z_{w_\zeta\cup\{\gamma(\zeta)\}}^a * F_\mu) &= \langle(\bar{N} \cap W_\zeta^a) \cup Z_{\{\gamma(\zeta)\}}^a\rangle * (\bar{N} \cap F_\mu) \\ &\cong \langle(\bar{N} \cap W_\zeta^a) \cup Z_{\{\gamma(\zeta)\}}^a\rangle * F_{\aleph_0}. \end{aligned}$$

We claim that $X \cap \bar{N}$ is a free basis over W^a_ζ of what it generates over W^a_ζ, namely $Z^a_{w_\zeta \cup \{\gamma(\zeta)\}} * F_{\aleph_0}$. Otherwise there were finite sets $X_0 \subseteq X$ and $Y_0 \subseteq W^a_\zeta$ such that $X_0 \cup Y_0$ satisfies an equation which does not follow from the laws of the variety and the equalities in $\mathrm{Diag}(W^a_\zeta)$. By elementarity we can find $Y_1 \subseteq W^a_\zeta \cap \bar{N}$ such that $X_0 \cup Y_1$ satisfies the same equation, a contradiction.

To prove (12), if a is replaced by c, then (12) is easily verified by using the free basis $\{c_\nu : \nu < \lambda\}$. But then using (10) and $K^c = K^a * K^b$ we easily finish.

Finally, as $\delta^* \in \mathcal{C}$ and δ^* has uncountable cofinality, by (3) we may choose $F_{\aleph_0} \subseteq \langle\{a_\nu : \nu \in \delta^*\}\rangle$ (i.e. an algebra isomorphic to $F_{\aleph_0}$) such that $M \cap F_{\aleph_0} = \emptyset$ and even $\langle M \cup F_{\aleph_0}\rangle = F_{\aleph_0} * M$. By (12) we conclude

$$W^a_{\zeta+1} = Z^a_{w_\zeta \cup \{\gamma(\zeta)\}} *_{F_{\aleph_0} * M} (F_{\aleph_0} * *_M \{Z^a_{\{\beta(\zeta,n)\}} : n \in \omega\}).$$

Moreover by construction (as $\beta(\zeta, n) \in S_2$), $Z^a_{\{\beta(\zeta,n)\}} \cong_M N$ for every $n \in \omega$. Hence $F_{\aleph_0} * *_M\{Z^a_{\{\beta(\zeta,n)\}} : n \in \omega\} \cong_M F_{\aleph_0} * *_M\{N : n \in \omega\}$. By assumption $F_{\aleph_0} * *_M\{N : n \in \omega\}/F_{\aleph_0} * M$ is free, and so clearly of rank $\aleph_0$. We conclude that $W^a_{\zeta+1} \cong_{W^a_\zeta} Z^a_{w_\zeta \cup \{\gamma(\zeta)\}} * F_{\aleph_0}$, and so $W^a_{\zeta+1}/W^a_\zeta$ is free by (11). ∎

References

[EkMe] P. C. Eklof and A. H. Mekler, *Almost Free Modules: Set-theoretic Methods*, North-Holland, Amsterdam, 1990.

[MeSh] A. H. Mekler and S. Shelah, *Almost free algebras*, Israel Journal of Mathematics **89** (1995), 237–259.

[Sh] S. Shelah, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel Journal of Mathematics **21** (1975), 319–349.